Stochastic approach for the subordination in Bochner sense

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It is possible to construct (cf [1] and [2]) a positive process $(Y_{\psi,t})$ indexed by Bernstein functions and time which, for fixed t, is Markovian with respect to the composition of Bernstein functions, and for fixed ψ, is the subordinator associated with the Bernstein function ψ. In the same manner a realization $(X_{Y_{\psi,t}})$ can be obtained of all the subordinated processes of a Markov process (X_t).

This probabilistic interpretation of the initial idea of Bochner allows to construct martingales and to apply stochastic calculus to questions related to symbolic calculus for operators semi-groups.

We study here a particular branch of the subordination process : The homographic branch which has the advantage of being connected with some works on positive infinitely divisible diffusions and brings on them a different point of view. Nevertheles, what follows would apply, except technical difficulties, to any branch of the subordination process.

It is easy to see that the functions

$$f_a(x) = \frac{xe^a}{1 + x(e^a - 1)}, \quad a \geq 0, x \geq 0$$

satisfy $f_a \circ f_b = f_{a+b}$ and are Bernstein functions (cf [3]) associated with the subordinators

$$Y_t^a = (e^a - 1) \sum_{k=1}^{N_{t/(1-e^a)}} E_k \tag{1}$$

where (N_t) is a standard Poisson process and the E_k's are i.i.d. exponential random variables independent of N. The Lévy measure of $(Y_{a,t})$ is therefore

$$\nu_a(dy) = \frac{1}{(e^a - 1)^2} e^{-y/(e^a-1)}\, dy \quad \text{ on } \mathbb{R}_+.$$

The relation $P_a u(x) = \mathbb{E}[u(Y_x^a)]$ for $u : \mathbb{R}_+ \mapsto \mathbb{R}_+$ defines a Markovian semi-group $(P_a)_{a\geq 0}$ with generator $Au(x) = xu'(x) + xu''(x)$ which is the transition semi-group of the diffusion $(Z_a)_{a\geq 0}$ solution to the sde

$$dZ_a = \sqrt{2Z_a} dB_a + Z_a da, \quad Z_0 = z \geq 0. \tag{2}$$

We now define a two parameters process $(Y_{a,t})_{a\geq 0, t\geq 0}$ by chosing for $a_1 < a_2 \cdots < a_n$ the joint law of the processes

$$(Y_{a_1,t})_{t\geq 0}, \ldots, (Y_{a_n,t})_{t\geq 0}$$

to be that of the process

$$\left(Y_t^{a_1}, Y_{Y_t^{a_1}}^{a_2-a_1}, \ldots, Y^{a_n-a_{n-1}}_{Y^{a_{n-1}-a_{n-2}}_{\ddots_{Y_t^{a_1}}}} \right)_{t\geq 0}$$

where $(Y_t^{a_1})_{t\geq 0}, (Y_t^{a_2-a_1})_{t\geq 0}, \ldots, (Y_t^{a_n-a_{n-1}})_{t\geq 0}$ are independent subordinators of type (1).

A version of the process $(Y_{a,t})$ may be chosen such that $Y_t = Y_{.,t}$ be right continuous and increasing with values in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ and with independent stationary increments, and for fixed t $(Y_{a,t})_{a\geq 0}$ has the same law as (Z_a) for $Z_0 = t$.

Because of the formula $\nu_a P_b = \nu_{a+b}$, we can construct the process $(Z_a)_{a>0}$ with the entrance rule (ν_a) what defines a positive σ-finite measure m on the space $\mathcal{C}(\mathbb{R}_+^*, \mathbb{R}_+)$ as the 'law' of the process (Z_a) under the entrance law (ν_a).

This measure m is the Lévy measure of the process $(Y_t)_{t\geq 0}$. This can be seen in the following way. Let μ be a positive measure with compact support on $\mathbb{R}_+$, let us put $< Y_t, \mu >= \int Y_{a,t}\, \mu(da)$. The relation

$$\mathbb{E}e^{-<Y_t,\mu>} = \exp \int (e^{\int Z_\alpha \mu(d\alpha)} - 1)\, dm \tag{3}$$

is easy to prove when μ is a weighted sum of Dirac masses by the computation

$$\mathbb{E}e^{-<Y_t,\lambda\varepsilon_a>} = \mathbb{E}e^{-\lambda Y_{a,t}} = e^{-tf_a(\lambda)} = e^{t\int (e^{-\lambda y}-1)\,\nu_a(dy)}$$

noting that ν_a is the law of Z_a under m. And for general μ (3) is obtained by weak limit.

The law of the subordinators $< Y_t, \mu >$, may be studied by using the fact that if we decompose μ into

$$\mu = 1_{[0,x]}.\mu + 1_{(x,\infty)}.\mu = \mu_1 + \mu_2$$

the following representation holds

$$< Y_t, \mu >=< Y_t, \mu_1 > + \int Y_{Y_{x,t}}^{b-x}\, \mu_2(db)$$

where $(Y_t^\beta)_{\beta\geq 0, t\geq 0}$ is independent of $(Y_{a,t}, a \leq x, t \geq 0)$.

One obtains for $\lambda \geq 0$,

$$\mathbb{E}\exp\{-\lambda < Y_t, \mu >\} = \exp\{-t(\frac{1}{2} - g'_x(0, \lambda))\}$$

where g is the positive solution decreasing in x of

$$\begin{cases} g''_{x^2} = & g.(\frac{1}{4} + \lambda.\mu) \\ g(0, \lambda) = & 1 \end{cases}$$

and if we put $\xi(x,\lambda) = \frac{1}{2} - \frac{g'_x(x,\lambda)}{g(x,\lambda)}$, the process

$$M_{a,t} = \exp\{-\xi(a,\lambda)Y_{a,t} + \xi(a,\lambda)t - \lambda \int_0^a Y_{\alpha,t}\,\mu(d\alpha)\}$$

is a two parameters martingale for the filtration of $(Y_{a,t})$.

A similar study may be performed with the family $g_a(x) = \frac{x}{1+ax}$ which corresponds to the diffusion

$$Z_a = z + \int_0^a \sqrt{2Z_b}\,dB_b. \tag{4}$$

wich is the square of a Bessel process of exponent 0, cf [4].

These results were obtained by Pitman and Yor [5], [6], by interpretating the process $Y_{a,t}$ of the case (4) as $\ell^a_{\tau_t}$ where ℓ^a_t is the family of brownian local times and $\tau_t = \inf\{s : \ell^0_s = t\}$, and using the Ray-Knight theorems. In this framework the measure m appears to be the image of the Ito measure of the Brownian excursions.

References

[1] N. Bouleau and O. Chateau "Le processus de la subordination", *C. R. Acad. Sc. Paris* t. 309, s1, 625-628, (1989).

[2] O. Chateau *Quelques remarques sur les processus à accroissements indépendants et la subordination au sens de Bochner* Thèse Univ. Paris VI, (1990).

[3] C. Berg and G. Forst, *Potential theory on locally compact groups*, Springer (1975).

[4] T. Shiga and S. Watanabe, "Bessel diffusions as a one parameter family of diffusion processes" *Z. f. Wahr.* 27, 37-46, (1973).

[5] J. Pitman and M. Yor, "A decomposition of Bessel bridges",*Z. f. Wahr.* 59, 425-457, (1982).

[6] J. Pitman and M. Yor, "Sur une décomposition des ponts de Bessel", Lect. N. in Math. 923, p.276 *et seq.*, (1982).